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Dibaryons from exceptional collections

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Abstract: We discuss aspects of the dictionary between brane configurations in del Pezzo geometries and dibaryons in the dual superconformal quiver gauge theories. The basis of fractional branes defining the quiver theory at the singularity has a K-theoretic dual exceptional collection of bundles which can be used to read off the spectrum of dibaryons in the weakly curved dual geometry. Our prescription identifies the R-charge R and all baryonic U(1) charges Q_I with divisors in the del Pezzo surface without any Weyl group ambiguity. As one application of the correspondence, we identify the cubic anomaly $\operatorname{tr} RQ_IQ_J$ as an intersection product for dibaryon charges in large-N superconformal gauge theories. Examples can be given for all del Pezzo surfaces using three- and four-block exceptional collections. Markov-type equations enforce consistency among anomaly equations for three-block collections.

Keywords: D-branes, AdS-CFT and dS-CFT Correspondence, Differential and Algebraic Geometry.

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1. Introduction, summary of results, and outlook

AdS/CFT dual pairs with $\mathcal{N} = 1$ supersymmetry have been much studied as natural generalizations of the original duality [1, 2, 3] between $\mathcal{N} = 4$ SYM and type-IIB strings on $AdS_5 \times \mathbf{S}^5$. After breaking of conformal invariance and supersymmetry, and combined with efforts at solving the strongly coupled worldsheets in small radius Anti-de Sitter space, they hold the promise of a holographic understanding of gauge theories describing the real world (see [4] for a recent review).

A convenient way of engineering conformal $\mathcal{N} = 1$ dualities is to place a stack of D3-branes at a conical Calabi-Yau threefold singularity [5]–[10]. To be specific, we will denote by $\mathbf{X}$ the Calabi-Yau cone over the five-dimensional Sasaki-Einstein manifold $\mathbf{Y}$, which in the class of examples we consider is itself a circle bundle over a smooth Kähler-Einstein surface $\mathbf{V}$ of positive curvature.[1] The theory on D3-branes at the tip of $\mathbf{X}$ is, at weak 't Hooft coupling, a quiver gauge theory, i.e., it has unitary gauge groups, bifundamental matter and a polynomial superpotential. At strong coupling, the theory is described by strings on $AdS_5 \times \mathbf{Y}$, with five-form flux on $\mathbf{Y}$. Breaking of conformal invariance can, in principle, be achieved on the gauge theory side by introducing fractional branes, and in the dual geometry by turning on additional fluxes leading to a warping of AdS and to a deformation of $\mathbf{Y}$. In practice, of course, these deformations are rather difficult to study, and have been explicitly realized only in a very small set of examples, most prominently the conifold itself [11]. In this paper, we will consider only the conformal case, but we believe that our results will be very useful for future non-conformal deformations of the duality.

1.1 Dibaryons

One of the classical tests of AdS/CFT involves the matching of the spectrum and algebra of BPS states on the two sides of the duality. One of the entries in this dictionary relates certain large-N non-perturbative states in the gauge theory, e.g.,

$$\epsilon_{i_1 i_2 \ldots i_N} \epsilon^{j_1 j_2 \ldots j_N} X^{i_1}_{j_1} X^{i_2}_{j_2} \cdots X^{i_N}_{j_N} \,, \tag{1.1}$$

where X^i_j is one of the bifundamental chiral matter fields in the quiver theory, with D-branes wrapping various cycles in the dual geometry, for example, a D3-brane wrapping a holomorphic curve in $\mathbf{V}$ together with the U(1) fiber of $\mathbf{Y}$. Following earlier literature [12], we will generically refer to these objects as dibaryons.

The name dibaryon comes from the fact that the definition (1.1) involves the antisymmetrization over fundamental indices of two gauge groups. Recall that the baryon, a more fundamental object obtained by antisymmetrizing over one gauge group, only exists when the theory is coupled to external charges [13]. In geometry, the baryon corresponds to a D5-brane wrapped on $\mathbf{Y}$ with fundamental strings attached to it by the presence of the RR flux. In general, other states can be obtained by antisymmetrizing over more than two gauge groups, and should perhaps be called "polybaryons". However, these objects can also be thought of as bound states of an equal number of baryons and anti-baryons (because the D3-brane is a bound state of a D5 and an anti-D5). In this sense, the denomination "dibaryon" is appropriate after all.

These dibaryons were first studied in [13] for the geometry $\mathbf{Y} = \mathbb{RP}^5$ which is dual to a supersymmetric SO$(2N)$ gauge theory. This gauge theory has no bifundamental matter but does have a field Φ_{ij} transforming in the adjoint of SO$(2N)$. The pfaffian Pf(Φ) is a gauge invariant operator formed by antisymmetrizing over $N/2$ copies of the Φ_{ij}. As the pfaffian consists of $N/2$ gluons, one expects it to have a mass of order N. The string coupling $\lambda \sim 1/N$, and it seems logical to associate the pfaffian, as it has a mass of order $1/\lambda$ to a wrapped D-brane in the dual geometry. In particular, the pfaffian is dual to a D3-brane wrapping a torsion cycle in the $\mathbb{RP}^5$.

[1] $\mathbf{V}$ has dimension 2 and we add an extra leg for each additional direction.

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Dibaryons in other simple geometries were studied soon after. The authors of [12], considered $\mathbf{Y} = \mathbf{S}^5/\mathbb{Z}_3$ and the dual gauge theory $\mathcal{N} = 1$ SU$(N)^3$ Yang Mills with nine bifundamental matter fields X_a, Y_b, and Z_c, three between each pair of SU(N) gauge groups. Dibaryons in $\mathbf{Y} = T^{1,1}$ were studied by [14]. The coset space $T^{1,1}$ is a level surface of the conifold. Here the gauge group is SU$(N)^2$ and there are two types of bifundamentals A_a and B_b where $a, b = 1, 2$. More elaborate examples of dibaryons have been considered since then. For example, Beasley and Plesser [15] give a detailed treatment of dibaryons for $\mathbf{Y}$ a U(1) bundle over the third del Pezzo surface. The AdS/CFT dictionary for dibaryons provides one of the few tests that AdS/CFT is a duality between string theory and gauge theory and not merely between supergravity and gauge theory.

Up until quite recently, work on dibaryons proceeded example by example. In a recent advance, [16] describes how to calculate the dibaryon mass for a general Sasaki-Einstein space $\mathbf{Y}$, given some information about the intersection of homology cycles in $H^2(\mathbf{V})$. In another intriguing recent paper [17], Intriligator and Wecht suggest that one should directly associate a divisor in $H^2(\mathbf{V})$ to each bifundamental field X^i_j at least in the case where all gauge groups are SU(N). Such an association would greatly improve our understanding of the dibaryon spectrum, allowing direct comparison between antisymmetric products of the X^i_j and the dual cycles wrapped by D3-branes. The genesis of the present paper was an attempt to make such an association more precise and to understand where it could come from.

Thus, our primary goal is to describe a scheme that allows a complete and unambiguous identification between dibaryons in the quiver theory and branes wrapped on holomorphic curves in the dual geometry. Along the way, we will uncover an interesting structure that we believe can be applied to a variety of other questions about this class of gauge/gravity dualities. In the remainder of the introduction, we will give an overview of our results. The details are split between the subsequent sections 2–7.

1.2 Gauge theories at threefold singularities and exceptional collections

Before the start, it is necessary to know the gauge theory. For orbifolds [5, 6] (where $\mathbf{Y} = \mathbf{S}^5/\Gamma$), the field content and superpotential can be derived at weak 't Hooft coupling from perturbative string theory using the methods of Douglas and Moore [18], and follows essentially from the representation theory of finite groups. For a certain limited class of other examples with non-spherical horizon, mainly toric del Pezzos, the gauge theories can be obtained by higgsing, or partial resolution, of orbifold singularities. These methods yield the gauge theories up to ambiguities related to "toric duality" [19, 20] — which can be understood in terms of Seiberg duality [15, 21].

One can also take a slightly different point of view to study the gauge theories at weak coupling and ask what happens when one resolves the singularity and follows the D-branes to the corresponding large volume Calabi-Yau manifold, an approach that has been developed in recent years in particular in [23]–[27]. This method has the advantage that it can in principle be applied also to non-toric geometries, such as the general del Pezzos. As we will review below, the classical theory of exceptional collections of bundles (or sheaves) appears naturally at large volume and is a useful tool for organizing the

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gauge theories on branes at threefold singularities. Exceptional collections also play an important role in the context of mirror symmetry, where they are mirror duals of certain exceptional branes in Landau-Ginzburg models. A small list of references is [22, 29, 30, 31]. In particular, exceptional collections and their duals shed an interesting light on the Cecotti-Vafa classification program of $\mathcal{N} = 2$ theories in two dimensions [32], and it is conceivable that there is a relation to our present work. The relevance of exceptional collections in understanding the geometric structure of gauge theories through gauge/gravity duality has been emphasized recently in [17, 22, 33]. Here, we will see that the collections are also useful for understanding the dibaryon spectra.

We will now try to explain the relevant ideas of [23]–[27] in a popular example, the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold. For a more pedagogical review, see [28]. We will review some material on exceptional collections in section 2.

As is well-known, the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold is continuously connected through variation of Kähler parameters to the large volume non-compact Calabi-Yau manifold $\mathcal{O}_{\mathbb{P}^2}(-3)$ which is the total space of the canonical bundle over the complex projective plane $\mathbb{P}^2$. We know from [18] that probe D3-branes on this Calabi-Yau experience an enhancement of gauge symmetry when the (closed string) Kähler modulus is tuned to the orbifold point. More precisely, the gauge group is enhanced from $\mathrm{U}(1)$ to $\mathrm{U}(1)^3$ and there are extra light fields X_a, Y_b, Z_c appearing in bifundamental representations $(a, b, c = 1, 2, 3)$. The superpotential is of the well-known cubic form $\epsilon^{abc} X_a Y_b Z_c$. It is customary to view the D3-brane as the bound state of three elementary constituents, the fractional branes. In fact, it is claimed that all possible D-branes on $\mathbb{C}^3/\mathbb{Z}_3$ can be obtained as bound states of these three fractional branes [24]. If this is true, and there are no antibranes involved, then all possible D-branes on the orbifold can be described as supersymmetric configurations in an $\mathcal{N} = 1$ field theory. As we move away from the orbifold point, supersymmetry is broken spontaneously [27]. In $\mathcal{N} = 2$ language (related to the previous language by three T-dualities in the spatial direction, whereby the D3-brane becomes a D0-brane in type IIA), the BPS central charges of the three fractional branes cease to be aligned.

One of the important facts in this context is the decoupling statement [27], which is based on the topological twisting of the theory, and essentially states that holomorphic information in the worldvolume theory of the D-branes does in fact not depend on the Kähler parameters. In particular, the chiral spectrum and the holomorphic superpotential (F-terms) can be computed at large volume in classical geometry. The Kähler parameters do, in the second step of finding supersymmetric vacua, only affect the D-terms in the worldvolume theory.

Let us make these statements precise in the example. As first derived using mirror symmetry [23], the three fractional branes (e_1, e_2, e_3) of $\mathbb{C}^3/\mathbb{Z}_3$ are related at large volume to the collection of three bundles

$$\mathcal{E} = (E_1, E_2, E_3) = (\mathcal{O}(-1), \overline{T^*(1)}, \mathcal{O}), \tag{1.2}$$

where $T^*(1)$ is the twisted cotangent bundle, and we have omitted introducing a notation for the fact that we want to extend these bundles over $\mathbb{P}^2$ to the Calabi-Yau. (All "bundles" in the following discussion are sheaves with support on the compact space, and never extend

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in the non-compact directions.) In (1.2), we have used the overbar notation to denote that we have actually obtained the antibrane related to this bundle.

As explained above, the chiral spectrum and the superpotential can now be easily computed from classical considerations. For instance, the massless fields between two branes defined by the bundles E_1 and E_2 are given by elements of the Dolbeault cohomology groups $H^{0,k}(E_1^* \otimes E_2) = \mathrm{Ext}^k(E_1, E_2)$. Here, the star denotes the dual bundle and should not be confused with the overbar in (1.2). For the collection in (1.2) one finds that there are precisely three chiral fields between each pair of bundles. The superpotential is computed by multiplication of sections to be the familiar cubic superpotential, and in this way, one has reproduced exactly the orbifold results.

What these classical considerations cannot give is the D-terms in the gauge theory. At large volume, the central charge

$$Z(E) = \int e^{-\omega}\,\mathrm{ch}(E)\sqrt{\mathrm{Td}(\mathbf{X})} \tag{1.3}$$

of a brane depends only on the dimensionality of the corresponding submanifold (and the GSO projection), and is independent of the gauge bundle. However, at small volume, the formula (1.3) receives corrections from worldsheet instantons, which are responsible for the "flow of gradings" that maps the collection (1.2) of two branes and one antibrane at large volume to the three fractional branes at the orbifold point with equal central charge [24, 27].

Let us note two remarkable facts about the gauge theory whose derivation from large volume considerations we have just reviewed. First, there are no matter fields in the adjoint representations of the gauge group, hence the gauge theory does not have a Coulomb branch. Second, all matter fields between two fractional branes are chiral. These two facts are precisely what makes the collection of bundles (1.2) "exceptional" in the mathematical sense. In the language of bundles, adjoint matter fields would arise from non-trivial morphisms from a bundle to itself; in other words, they would correspond to deformations of the bundle. The chirality of the matter means that, for fixed $i \neq j$, of all possible groups $\mathrm{Ext}^k(E_i, E_j)$, at most one is non-zero, and in that case $\mathrm{Ext}^k(E_j, E_i)$ vanishes for all k.

Having reviewed the central ideas, we are now in a position to refer to the recent work of Wijnholt [33] for applications to the del Pezzo surfaces dP_n which will be the focus of our interest in the subsequent sections. In particular, gauge theories have been derived in [33] for branes at the tip of cones over toric and non-toric del Pezzos in a uniform manner, starting from exceptional collections on dP_n. It is worthwhile to point out that one is implicitly assuming that there actually is a point in the Kähler moduli space at which the central charges of the exceptional collections all align. The existence of such a point does not follow from any results known to us. (Mirror symmetry is of no help here, because the mirrors of the general del Pezzos are not known.) In other words, it is not clear to us at what point, if any, in the moduli space of $\mathbf{X} = dP_n(K)$, the gauge theory actually lives. In what follows, we will simply assume that there is such a point and that semi-classically this point corresponds to a singular cone over the del Pezzo.

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1.3 Main results

In this paper, we consider AdS/CFT dual pairs of theories which are obtained by choosing for $\mathbf{V}$ a del Pezzo surface and for $\mathbf{X}$ the complex cone over $\mathbf{V}$. On the gauge theory side, we have the large-N limit of the quiver gauge theories, which one can derive from an exceptional collection on $\mathbf{V}$ as we have briefly reviewed in the previous subsection. We would like an identification of the dibaryon operators in the field theory with branes in $\mathbf{Y}$. We claim that the spectrum of dibaryons follows in a natural way from an exceptional collection of bundles on $\mathbf{V}$ "dual" to the collection defining the quiver theory. We explain this "duality" in detail in section 4. In the example $\mathbf{V} = \mathbb{P}^2$, the dual collection is

$$\mathcal{E}^\vee = (E_3^\vee, E_2^\vee, E_1^\vee) = (\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)) . \tag{1.4}$$

Our proposal associates to each of the bifundamental fields in the quiver a (fractional) homology class in $\mathbf{V}$ which is the difference in the first Chern classes of the bundles on the corresponding nodes in the dual collection. Just as a collection of bifundamentals can antisymmetrize to form a gauge invariant dibaryon, a collection of these fractional homology classes can add up to an (integral) curve in $\mathbf{V}$ which a D3-brane wraps together with the U(1) fiber of the fibration $\mathbf{Y} \to \mathbf{V}$.

For $\mathbb{C}^3/\mathbb{Z}_3$, (1.4), our prescription simply associates with each of the bifundamental fields X_a, Y_b, Z_c the hyperplane class in $\mathbb{P}^2$, thus reproducing the results of [12]. For example, the fields X_a run from node 3 to node 2 in figure 1, which in the dual collection (1.4) correspond to the bundles $\mathcal{O}$ and $\mathcal{O}(1)$, respectively. Thus, we associate with the dibaryon constructed out of X_a the hyperplane divisor $H = c_1(\mathcal{O}(1)) - c_1(\mathcal{O})$. The careful reader will suspect that our ordering and sign conventions must be quite complicated, and may also wonder about the field Y, which would get the divisor $-2H$ according to the rules as stated up to now. Indeed, a number of refinements will be necessary, and we will explain them in section 4.

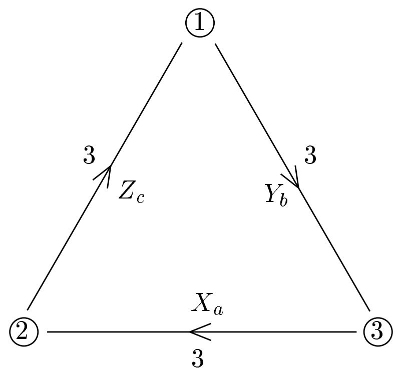

Figure 1: Quiver for $\mathbb{P}^2$.

This identification of baryon spectra is the central result of our paper. For completeness, let us also mention here a number of side results that are of interest in their own right.

Our $\mathcal{N} = 1$ superconformal gauge theories enjoy a number of global U(1) symmetries under which the dibaryons carry charge. On the AdS side, these symmetries are gauge symmetries arising from Kaluza-Klein compactification on $\mathbf{Y}$. Our proposal includes a precise mapping of these U(1)'s under the duality.

The U(1) R-symmetry, for example, arises from the U(1) isometry of $\mathbf{Y}$, and the R-charge of a dibaryon B can be computed by intersecting the corresponding curve C with the canonical divisor in $\mathbf{V}$, according to the formula [16, 17]

$$R(B) = \frac{2N(-K) \cdot C}{K^2} , \tag{1.5}$$

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where K is the canonical class of $\mathbf{V}$. For the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold, we know that the anomalous dimensions of the X, Y, and Z fields vanish, so their R-charge is $2/3$. Antisymmetric products of N bifundamentals will have R-charge $2N/3$, and formula (1.5) reproduces this result ($K = -3H$ for $\mathbb{P}^2$).

Other U(1) symmetries, so-called "baryonic" U(1)'s, arise on the AdS side from reducing the RR 4-form on three-cycles in $\mathbf{Y}$, which project to divisors in $\mathbf{V}$. Our prescription for identifying dibaryons with curves allows a map from each baryonic charge Q_I to a specific divisor in $\mathbf{V}$. One interesting point is that we are also able to identify the gauge theory object that computes the intersection of divisors in $\mathbf{V}$. This intersection product is simply the cubic anomaly $\operatorname{tr} RQ_IQ_J$. We will discuss these baryonic U(1)'s and their intersection product in sections 3 and 4.

A different class of results in our paper — which are not all new — concerns algebraic conditions on anomalies in quiver gauge theories related to del Pezzos. We recall that, generally, vanishing of the chiral anomaly imposes certain restrictions on possible brane wrapping numbers. For the n-th del Pezzos dP_n [22], the restriction identifies an $n+1$ dimensional lattice of allowed brane wrappings, including both D3 and D5-branes. To find the D3-brane, one must identify which of these configurations allows a vanishing NSVZ beta function, or equivalently, vanishing of the R-charge anomaly at each node in the quiver. The n D5-branes wrapped on vanishing cycles break conformal invariance and do not allow an R-symmetry. There is a canonical one-to-one correspondence between the n D5-branes and the baryonic U(1)'s mentioned above (see also [17]).

Our strongest results are for the three-block exceptional collections. Three-block collections give rise to quivers with three blocks of nodes. Between two nodes in a block, there are no arrows. Karpov and Nogin [34] have studied three-block collections over del Pezzo surfaces and have shown that they are classified by certain diophantine equations generalizing the Markov equation

$$x^2 + y^2 + z^2 = 3xyz\,. \tag{1.6}$$

Eq. (1.6) is known [22] to classify all possible gauge theory quivers for the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold. In particular, the x, y, and z are the ranks of the three gauge groups. Karpov and Nogin [34] find a generalized Markov equation

$$\alpha x^2 + \beta y^2 + \gamma z^2 = \sqrt{K^2\alpha\beta\gamma}\,xyz \tag{1.7}$$

which classifies the three-block exceptional collections for del Pezzos. There are α nodes in the first block, β nodes in the second, and γ in the third and x, y, z are the ranks as before. Through classifying three-block collections, this equation (1.7) also classifies a large set of quivers for the corresponding gauge theories.

For three-block collections over del Pezzos, we show how the generalized Markov equation (1.7) arises as the consistency condition between the NSVZ beta function and R-charge of the superpotential.[2] We are also able to compute various cubic 't Hooft anomalies explicitly. For example, AdS/CFT predicts that $\operatorname{tr} R^3 = 24N^2/K^2$. We are able to confirm this relation for all three-block collections. We also check the maximizing-a principle of [36],

[2]This connection was made for $\mathbb{P}^2$ and (1.6) in [35].

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i.e., that $\operatorname{tr} R^2 Q_I = 0$ for a superconformal field theory and that $\operatorname{tr} RQ_IQ_J$ has all negative eigenvalues, where Q_I are the baryonic U(1)'s mentioned above. Finally, we are able to show that $\operatorname{tr} Q_IQ_JQ_K = 0$ for three-block collections.

To make the presentation less abstract, we will illustrate the working of our machinery in a number of examples. We work out dibaryon spectra for four block quivers for $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ blown up at a point, and dP_2. We give an extensive treatment of quivers that arise from three block collections. Finally, we work out all 240 smallest dibaryons for a three-block colllection for dP_8.[3]

1.4 Outlook

We will see in this paper that exceptional collections, which appear naturally in brane-geometric engineering of $\mathcal{N} = 1$ gauge theories, are also useful for understanding the baryon spectra in AdS/CFT dual pairs arising from del Pezzo geometries. The fact that the above identifications are highly non-trivial give us reason to expect that our results will shed light on a number of other related questions, for example regarding non-conformal deformations of AdS/CFT. We hope that the unambiguous mapping of gauge theory charges to cycles in the geometry will help us understand how to add fluxes in the dual geometry to break conformal invariance in a controlled way.

We also feel that our results shed new light on the relation between Seiberg duality and all its cousins (see [22, 37, 38, 39]). It was argued in [22] that Seiberg duality is geometrically realized as mutations of the bundles defining the gauge theory or as Picard-Lefshetz transformations in the mirror geometry. Our results point to the importance of the exceptional collection (1.4) dual to the bundles defining the gauge theory. Indeed, Seiberg duality in the quiver gauge theory is (in some cases) geometrically realized as a sequence of mutations on this dual exceptional collection.

Finally, let us mention the extremely large class of largely unstudied examples where $\mathbf{X}$ is a generalized conifold or more generally the cone over a log-del Pezzo. In these cases, the associated surface $\mathbf{V}$ is singular, but the cone $\mathbf{X}$ is smooth except at the tip, and very interesting quiver gauge theories emerge, for example quivers identical to the ADE Dynkin diagrams [40]. Intersections of curves in $\mathbf{V}$ can give rise to rational numbers, and it's not clear how this rationality affects the charges of the dibaryons. For the generalized conifolds, the quivers are non-chiral, involving equal numbers of arrows in both directions between nodes. It would be interesting to see how to generalize the notion of exceptional collections to these non-chiral examples. Exceptional collections as currently understood seem to allow only for chiral quiver theories.

2. Exceptional collections and helices

As we have mentioned in the introduction, the notion of an exceptional collection of sheaves is a very natural mathematical concept in the context of D-branes. Among the recent physics literature, let us mention [17, 22, 30, 33]. The standard mathematical reference

[3] Martijn Wijnholt has also recently used the three-block exceptional collections of [34] for deriving quivers for dP_7 and dP_8.

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is [41]. For the string theorist, an exceptional collection is simply a set of elementary "rigid" branes generating all BPS configurations of the theory by bound state formation. We have here summarized a few of the basic mathematical definitions to fix notation, but readers may want to skip the rest of this section.

Let $\mathbf{V}$ be a complex Fano variety. A sheaf E over $\mathbf{V}$ is called exceptional if $\mathrm{Ext}^0(E,E) = \mathbb{C}$ and $\mathrm{Ext}^k(E,E) = 0$ for $k > 0$. An ordered collection $\mathcal{E} = (E_1, E_2, \ldots, E_m)$ of sheaves is called exceptional if each E_i is exceptional and if, moreover, for each pair E_i, E_j with $i > j$, we have $\mathrm{Ext}^k(E_i, E_j) = 0$ for all k and $\mathrm{Ext}^k(E_j, E_i) = 0$ except possibly for a single k.

For two sheaves E and F on $\mathbf{V}$, we have the generalized Euler character

$$\chi(E,F) = \sum_i (-1)^i \dim \mathrm{Ext}^i(E,F)\,, \tag{2.1}$$

which by the Hirzebruch-Riemann-Roch theorem can be rewritten as

$$\chi(E,F) = \int_{\mathbf{V}} \mathrm{ch}(E^*)\,\mathrm{ch}(F)\,\mathrm{Td}(\mathbf{V})\,, \tag{2.2}$$

where $\mathrm{ch}(E)$ is the Chern character of the sheaf E. The bilinear form χ is non-degenerate on (the torsion-free part of) the K-theory $K_0(\mathbf{V})$.

It is easy to see from the definition that for an exceptional collection $\mathcal{E} = (E_1, E_2, \ldots, E_m)$, the matrix S, with entries

$$S_{ij} = \chi(E_i, E_j)\,, \tag{2.3}$$

is upper triangular with ones on the diagonal. Hence S is invertible, and it follows that the maximal length of an exceptional collection is $m = \dim K_0(\mathbf{V})$. An exceptional collection which generates the derived category of coherent sheaves on $\mathbf{V}$ is called complete.

To be specific, we consider the del Pezzo surfaces, which are at the center of interest in this paper. The n-th del Pezzo surface dP_n has non-trivial Betti numbers $b_0 = 1$, $b_2 = n+1$. Hence, $\dim K_0(dP_n) = n+3$, and the maximal length of an exceptional collection is $n+3$. The Chern character of a sheaf $\mathrm{ch}(E) = (r(E), c_1(E), \mathrm{ch}_2(E))$ is given by $n+3$ "charges": the rank $r(E)$, the first Chern class $c_1(E) = \mathrm{ch}_1(E) \in H^2(dP_n, \mathbb{Z})$, and the second Chern character $\mathrm{ch}_2(E)$. In components, the Euler character reads

$$\begin{aligned}\chi(E,F) \;&=\; r(E)r(F) + \frac{1}{2}\left(r(E)\deg(F) - r(F)\deg(E)\right) + \\ &\quad + r(E)\,\mathrm{ch}_2(F) + r(F)\,\mathrm{ch}_2(E) - c_1(E)\cdot c_1(F)\,,\end{aligned} \tag{2.4}$$

which can easily be derived from (2.2) using $\mathrm{Td}(dP_n) = 1 - \frac{K}{2} + H^2$, where K is the canonical class and H is the hyperplane, with $\int_{dP_n} H^2 = 1$. Also, the degree $\deg(E) = (-K)\cdot c_1(E)$.

If $\mathcal{E}$ is an exceptional collection, one obtains new exceptional collections by so-called mutations. There are left and right mutations, so-called because of the way they affect the ordering of the sheaves,

$$L_i:\;\; (\ldots, E_{i-1}, E_i, E_{i+1}, \ldots) \to (\ldots, E_{i-1}, L_{E_i}E_{i+1}, E_i, \ldots)\,, \tag{2.5}$$

$$R_i:\;\; (\ldots, E_{i-1}, E_i, E_{i+1}, \ldots) \to (\ldots, E_{i-1}, E_{i+1}, R_{E_{i+1}}E_i, \ldots)\,. \tag{2.6}$$

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Here, $L_{E_i}E_{i+1}$ and $R_{E_{i+1}}E_i$ are defined by short exact sequences, whose precise form depends on which of the $\mathrm{Ext}^k(E_i, E_{i+1})$ is non-zero. For exceptional collections over dP_n, only Ext^1 and Hom are nontrivial [42]. For these mutations, if for an exceptional pair (E, F), we have $\mathrm{Ext}^0(E, F) = \mathrm{Hom}(E, F) = \mathbb{C}^c$, where $c = \chi(E, F)$, we can consider the canonical mappings

$$\mathrm{Hom}(E, F) \otimes E \to F\,, \tag{2.7}$$

$$E \to \mathrm{Hom}(E, F)^* \otimes F\,, \tag{2.8}$$

and left and right mutations are defined as kernel or cokernel of these mappings. So, if (2.7) has a kernel, we have

$$0 \to L_E F \to E^{\oplus c} \to F \to 0\,. \tag{2.9}$$

This particular mutation is called a division. The precise form of all the other mutations can be found, e.g., in [41].

We note that mutations operating on exceptional collections as in (2.6) satisfy the braid group relations

$$\begin{aligned} L_i R_{i+1} &= R_{i+1} L_i = 1\,, \\ L_{i+1} L_i L_{i+1} &= L_i L_{i+1} L_i\,. \end{aligned} \tag{2.10}$$

Mutations are one of the important checks of the relation, via Mirror Symmetry, between the theory of exceptional collections and Picard-Lefschetz theory for the singularities defining the associated mirror Landau-Ginzburg theories [29, 30]. Moreover, we mention that in the context of constructing $\mathcal{N} = 1$ gauge theories from D-branes, a certain subclass of mutations is related to Seiberg duality [22, 37, 43].

The most important aspect for us is the change in charges, which can be written as

$$\begin{aligned} \mathrm{ch}(L_E F) &= \pm(\mathrm{ch}(F) - \chi(E, F)\,\mathrm{ch}(E))\,, \\ \mathrm{ch}(R_F E) &= \pm(\mathrm{ch}(E) - \chi(E, F)\,\mathrm{ch}(F))\,, \end{aligned} \tag{2.11}$$

where the sign is chosen such that the rank of the mutated bundle is positive.

In fact, in the context of D-branes in string theory, it is more natural to allow also negative charges (which is simply an antibrane, the formal inverse of a bundle with positive rank). One then defines an associated left and right mutation of signed bundles L^D and R^D, which at the level of charges leads to the selection of the plus sign in (2.11) in all cases. (The superscript D refers to the derived category of coherent sheaves, to which the notion of exceptional collection is readily extended. We need these derived categories only implicitly here.) In those cases where Seiberg duality is related to a sequence of mutations, one has to take into account additional signs to keep the ranks of the gauge groups positive, see below.

Finally, let us consider a bi-infinite extension of an exceptional collection of sheaves $\mathcal{E} = (E_1, E_2, \ldots, E_m)$, defined recursively by

$$\begin{aligned} E_{i+m} &= R_{E_{i+m-1}} \cdots R_{E_{i+1}} E_i \\ E_{-i} &= L_{E_{1-i}} \cdots L_{E_{m-1-i}} E_{m-i} \qquad i \geq 0\,. \end{aligned} \tag{2.12}$$

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This infinite collection $\mathcal{H} = (E_i)_{i\in\mathbb{Z}}$ is called a helix of period m if

$$E_i = E_{m+i} \otimes K \qquad \forall i \in \mathbb{Z}\,, \tag{2.13}$$

where K is the canonical bundle of $\mathbf{V}$. Any subcollection of $\mathcal{H}$ of the form $(E_{i+1}, E_{i+2}, \ldots, E_{i+m})$ is exceptional and is called a foundation for $\mathcal{H}$. An interesting property of exceptional collections over Fano varieties (and del Pezzos in particular) is that an exceptional collection is complete (i.e., generates the derived category), if and only if it is the foundation of a helix.

3. Quiver theories from exceptional collections

In this section we summarize the construction of the quiver theory from an exceptional collection, discuss some properties of anomalies, and explain the general construction of dibaryons in these field theories.

As we have outlined in the introduction (see also [33]), we imagine starting from an exceptional collection $\mathcal{E} = (E_1, E_2, \ldots, E_m)$ on the surface $\mathbf{V}$ of interest. We extend this collection by zero to the total space of the canonical bundle $\mathbf{X} = \mathbf{V}(K)$ over $\mathbf{V}$, which is Calabi-Yau, and we then follow this collection to the "orbifold point" in the moduli space of $\mathbf{X}$. The collection $\mathcal{E}$, in particular the grading, is chosen such that at the orbifold point, all E_i correspond to mutually supersymmetric branes. We can then try to represent any brane B on $\mathbf{X}$ in a supersymmetric gauge theory, which is obtained as follows.

We start by drawing a quiver diagram, one node for each exceptional sheaf/fractional brane in the collection, and an arrow between two nodes i and j whenever there is a non-zero $\mathrm{Ext}^k(E_i, E_j)$. Our convention for the direction of the arrow is that it points from i to j if $\chi(E_i, E_j)$ is positive. We note that this sign depends not only on whether the arrow arises from an Ext^1 or Hom at large volume, but also on the gradings that we were forced to chose. It is sometimes convenient to think of an arrow as an ordered pair $a = (ij)$, and to introduce the notation $t(a) = i$ for the node at the tail of a and $h(a) = j$ for the node at the head of a. We will sometimes also use $w_a = \chi(E_i, E_j) \geq 0$ to denote the multiplicity or "weight" of the arrow $a = (ij)$.

Given the charge of any brane B that we want to describe in gauge theory, we can decompose its charge $\mathrm{ch}(B)$ in terms of the charges of the E_i,

$$\mathrm{ch}(B) = \sum_i d^i\ \mathrm{ch}(E_i)\,, \tag{3.1}$$

where we assume that all d^i are positive. We then associate in the gauge theory a gauge group of rank d^i to each node of the quiver, and chiral matter fields in bifundamental representations for each arrow (with chirality determined by the direction of the arrow). We will denote these fields by X_a or by X_{ij} for an arrow a from i to j.[4]

The other piece of information we need is the superpotential. At large volume, the superpotential encodes the relations between the morphisms from one exceptional bundle

[4] We will generally suppress flavor indices for arrows with multiplicity, because we will never break the non-abelian part of the flavor symmetry.

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to the other. Since the superpotential is independent of Kähler moduli, the relations between the chiral matter fields at the orbifold point will be the same as the ones obtained at large volume. For example, if $X_a \in \mathrm{Hom}(E_1, E_2)$, $Y_b \in \mathrm{Hom}(E_2, E_3)$, appear with multiplicities, then the composition of maps $Y \circ X$ leads to maps from E_1 to E_3 which need not all be independent, and the relations can be expressed by elements $Z \in \mathrm{Hom}(E_1, E_3)$. This relation between three chiral fields is encoded in a cubic term in the superpotential. See [22, 33] for additional details and examples. We note that in the quiver diagram, a superpotential term of order r is associated to a closed loop of arrows $(a_1, a_2, \ldots, a_r)$ visiting the nodes $(i_1, i_2 \ldots, i_r)$, and by gauge invariance must be of the form $W_{a_1,a_2,\ldots,a_r} = \mathrm{tr}\, X_{a_1} X_{a_2} \cdots X_{a_r} = \mathrm{tr}\, X_{i_1 i_2} X_{i_2 i_3} \cdots X_{i_r i_1}$, with appropriately contracted color indices.

3.1 Anomalies and non-conformal deformations

So far our description has been purely classical. We, however, are interested in studying strongly coupled gauge theories, and we have to make sure that our theories make sense at the quantum level. Cancellation of chiral gauge anomalies is equivalent to the condition that at each node, the number of matter fields in the fundamental and anti-fundamental be equal to each other. In terms of the matrix S, with entries $S_{ij} = \chi(E_i, E_j)$, this condition on the dimension vector $d = (d^1, d^2, \ldots, d^m)^t$ reads

$$\mathcal{I}\, d = (S - S^t)\, d = 0\,. \tag{3.2}$$

Given that $\mathcal{I} = S - S^t$ is the intersection form on the Calabi-Yau space X, the equation (3.2) has the interpretation that from the closed string perspective, we are only allowed to wrap cycles that do not intersect any other compact cycle, because otherwise the flux sourced by the brane has nowhere to go [22]. For example, in the del Pezzo case, we have one compact 4-cycle, $n + 1$ compact 2-cycles, and one compact 0-cycle in the geometry, which gives rise to $n + 1$ linearly independent choices for wrapping branes. In other words, $\mathcal{I}$ has rank 2.

How about conformal invariance? As is well-known, conformal invariance of an $\mathcal{N} = 1$ supersymmetric gauge theory is tied to the existence of an anomaly-free U(1) R-symmetry. In other words, to guarantee conformal invariance, we require that there be an assignment of R-symmetry charge $R(X_a)$ to each chiral multiplet X_a, such that at each node i, the NSVZ beta functions vanish,

$$2d^i + \sum_a w_a (R(X_a) - 1) \big(\delta^i_{t(a)} d^{h(a)} + \delta^i_{h(a)} d^{t(a)}\big) = 0 \qquad \text{for every node } i\,, \tag{3.3}$$

and that moreover, each term in the superpotential have R-charge 2

$$R(X_{i_1 i_2}) + R(X_{i_2 i_3}) + \cdots + R(X_{i_r i_1}) = 2 \qquad \text{for every loop } i_1, i_2, \ldots, i_r \text{ in the superpotential.} \tag{3.4}$$

In (3.3), $R(X_a)$ is the R-charge of the bottom component. Eqs. (3.3) and (3.4) are a system of inhomogeneous linear equations on the R-charges of the X_{ij}. Existence of a solution puts certain constraints on the possible numbers d^i. For the del Pezzos, we expect from string theory that out of the $n + 1$ possible brane wrappings, only the regular D3-brane

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at the orbifold will respect conformal symmetry, and all fractional branes will break it. In other words, we expect that out of the $n+1$ solutions of (3.2), exactly one of them will allow for an R-symmetry.

In general, there will also be other anomaly-free U(1) flavor symmetries. We can distinguish "mesonic" U(1) symmetries that do not assign the same charge to each of the several fields corresponding to one given arrow, in other words, that do not commute with the non-abelian part of the flavor symmetries [15], from "baryonic" U(1) symmetries that do commute with the non-abelian part of the flavor symmetries.[5] Baryonic U(1) charges, which we denote by Q_I, have to satisfy the homogeneous versions of (3.3) and (3.4),

$$\begin{aligned} \sum_a w_a Q_I(X_a)\left(\delta^i_{t(a)} d^{h(a)} + \delta^i_{h(a)} d^{t(a)}\right) &= 0 \qquad \text{for every node, and} \\ Q_I(X_{i_1 i_2}) + Q_I(X_{i_2 i_3}) + \cdots + Q_I(X_{i_r i_1}) &= 0 \qquad \text{for every loop.} \end{aligned} \tag{3.5}$$

We claim that the number of these baryonic charges is in general exactly one less than the number of solutions to eq. (3.2), and that moreover, for given choice of dimension vector d_* satisfying (3.2), the baryonic U(1)'s are in one-to-one correspondence with the other solutions of (3.2). To show this, let us define a complete basis of charges Q_a that commute with the non-abelian part of the flavor symmetries by $Q_a(X_b) = \delta_{ab}$. Any solution of (3.5) can be written as a combination of the Q_a, $Q_I = \sum_a q_I^a Q_a$. From the condition that the superpotential have charge 0, and assuming that there are enough independent terms in the superpotential, we infer that Q_I must give charge 0 to any field consisting of an incoming arrow at some node followed by an arrow coming out of that node. In other words, Q_I must be a linear combination of the charges Q_i that assign charge $+1$ to each arrow going into the node i and -1 to each outgoing arrow, $Q_I = \sum_i q_I^i Q_i$ with

$$Q_i = \sum_a (\delta_{i,h(a)} - \delta_{i,t(a)}) Q_a\,. \tag{3.6}$$

Using this definition, it is easy to see that the first condition in (3.5) becomes

$$\sum_j \mathcal{I}_{ij} q_I^j d_*^j = 0 \qquad \text{for every node } i. \tag{3.7}$$

Thus, if we denote by $d_I = (d_I^i)$ the solutions of (3.2), we can write the solutions to (3.5) as

$$q_I^i = \frac{d_I^i}{d_*^i}\,, \tag{3.8}$$

assuming that for our selected dimension vector d_*^i is non-zero for all i. We also note that for $d_I = d_*$, the corresponding U(1) is trivial, because it assigns charge 0 to all fields. This justifies the claim we made at the beginning of this paragraph.

The condition that d^i must be non-zero for all i for the baryonic charges to exist has a tempting interpretation based on the expectation about the RG behavior of our

[5] The fact that the R-symmetry behaves like the baryonic U(1)'s follows from the maximizing-a principle of [36].

theories. Let us assume that the conformal theory (stemming from D3-branes at a point) has all d^i positive. As we have just seen, the baryonic U(1)'s in this theory are in one-to-one correspondence with the possible non-conformal deformations of the theory [17]. Following [11], we can make a non-conformal deformation by adding fractional branes, and the deformation will be controllable if the number of fractional branes is much smaller than the number of regular D3-branes. Controllable means in particular that all d^i remain positive and the baryonic charges still exist, although they now slightly differ from the conformal case. The theory will then start flowing with the scale, and it is natural to expect a cascade similar to the one of Klebanov and Strassler [11]. In particular, the number of D3-branes will decrease along the cascade, until one eventually reaches a confining theory, where some of the d^i become zero, and the baryonic U(1)'s might disappear. On the gravity side, the RG flow and cascade correspond to a warping of the geometry, with sizes of various cycles in $\mathbf{Y}$ depending on the radial direction. Ultimately, confinement in the gauge theory is expected to correspond to a deformation of the singularity, with certain cycles smoothly shrinking to zero size in the geometry. In other words, the condition that the d^i must be positive for the baryonic U(1)'s to exist is related to the existence of cycles in the level surfaces of the dual geometry, and some of the baryonic U(1)'s will disappear in the IR when the gauge theory confines. In fact, it is shown in [22] that deformation of the geometry is possible precisely because some of the d^i vanish for certain non-conformal deformations of the theory involving a small number of regular D3-branes.

Let us return to the conformal case. If for some choice of d, there is an assignement of R-charge that satisfies (3.3) and (3.4), then this solution will not be unique because of the non-trivial solutions of (3.5). To fix this ambiguity, it is natural to first look for additional discrete symmetries of the quiver diagram that fix the R-charges of certain matter fields to be equal. However, in general these constraints do not suffice. Luckily, Intriligator and Wecht have recently shown [36] that there is a completely general method to fix the ambiguity. The prescription is that the exact superconformal R-charge is, among all solutions of (3.3) and (3.4), distinguished by the fact that it maximizes a, which is one of the central charges of the conformal algebra of the SCFT [44, 45],

$$a = \frac{3}{32}\big(3\,\mathrm{tr}\,R^3 - \mathrm{tr}\,R\big) = \frac{3N^2}{32}\bigg(\sum_a w_a d^{h(a)} d^{t(a)}\big[3(R(X_a)-1)^3 - (R(X_a)-1)\big] + \\ + 2\sum_i (d^i)^2\bigg)\,, \tag{3.9}$$

where the second term is the contribution from the gauginos. As we have mentioned, in some cases the exact R-symmetry is uniquely fixed by symmetries, but this is not true for an arbitrary quiver in which not all discrete symmetries are manifest. In the subsequent sections, we have made frequent use of the maximizing-a principle to determine the R-charge.

3.2 Dibaryons in quiver gauge theories

From now on, we will consider the conformal case only. This means that at each node i, we put a gauge group of rank Nd^i, where the vector $(d^1, d^2, \ldots, d^{n+3})^t$ is the (unique) null

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vector of $\mathcal{I}$ that admits an R-symmetry, and N is large. Our goal is to provide certain kinematical tests of $\mathcal{N}=1$ AdS/CFT by matching BPS observables on the two sides of the duality. At weak coupling, the observables in question are gauge invariant combinations of chiral operators. One possibility are traces. These are perturbative also in the large-N expansion, and correspond on the closed string side to gravitons or other supergravity fields [3]. On the other hand, determinant-like states, such as (1.1), are non-perturbative at large N (because their mass scales as one over the coupling constant) and correspond on the closed string side to branes.

For a convenient description of general dibaryon operators, let us for the moment just keep the vector space structure of the quiver, as we would do for the purposes of solving the F-terms or in the mathematical setting. In other words, we associate to each node i a vector space V_i of dimension Nd_i, and to each chiral field a linear map $X_{ij}: V_i \to V_j$. These linear maps furnish a representation of the path algebra $\mathcal{A}$ of the quiver diagram. We recall that the algebra structure means that products of paths are non-zero whenever arrows line up head to tail, and the associated map is simply a composition of maps between vector spaces. We can also consider sums of maps in the usual way.

Let us then consider an arbitrary element of the path algebra $A \in \mathcal{A}$

$$A : V_t \mapsto V_h \,, \tag{3.10}$$

where

$$V_t = \oplus_i t^i V_i \qquad V_h = \oplus_i h^i V_i \tag{3.11}$$

are the vector spaces at the "tail" and "head" of A, respectively. Note that A need not consist of only one path, and that some of the vector spaces V_i can appear with non-trivial multiplicity t^i, h^i in the tail and head of A. Moreover, we note that a non-trivial multiplicity $w_a > 1$ of some arrow a will give rise to a collection of different maps in general (unless there are relations from the superpotential).

A useful invariant, which we call the "rank" of A, is the difference in dimension of the vector spaces at tail and head,

$$r(A) = \dim(V_h) - \dim(V_t) = \sum_i N d^i (h^i - t^i) \,. \tag{3.12}$$

In gauge theory, $r(A)$ counts the number of uncontracted fundamental minus antifundamental indices in each gauge group. In order to form a gauge invariant operator out of A, we have to contract these open indices with external charges and take an antisymmetric combination over all the remaining ones. In the special case that $r(A)=0$, we can simply define this antisymmetrized product, the dibaryon, as the determinant of the linear map associated with A,[6]

$$B(A) = \det A \,. \tag{3.13}$$

We depict a few examples of dibaryons constructed along these lines in figure 2.

[6] One might worry that the determinant of a map between different vector spaces is not well-defined. However, we are here interested in gauge theory, so we will need a hermitian metric on each V_i. We can

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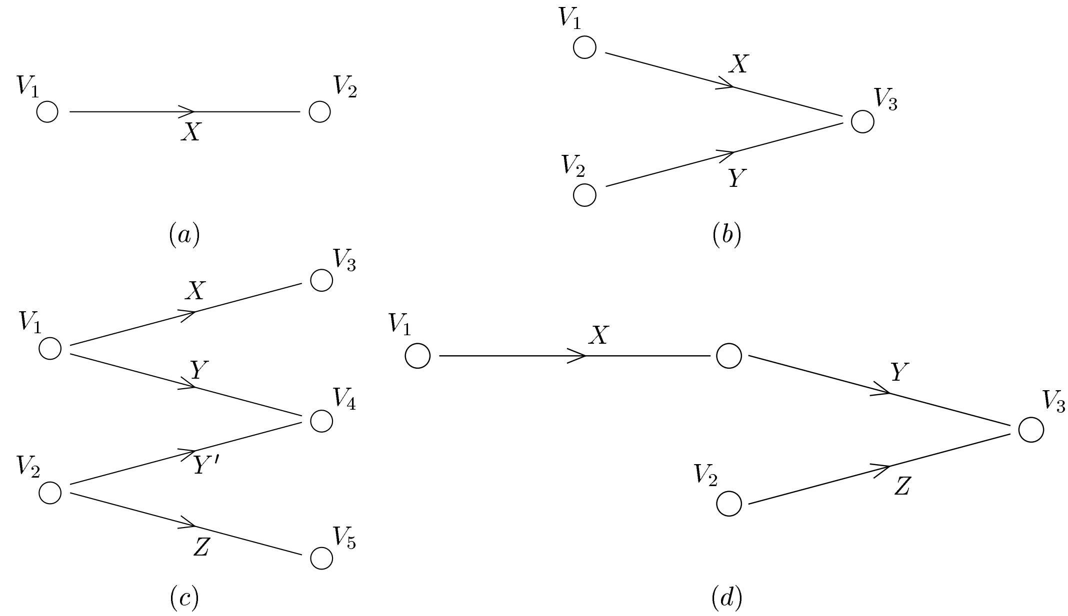

Figure 2: Examples of subquivers that can be used to construct dibaryon operators in the field theory.

In case (a), we have

$$\dim V_1 = \dim V_2 = N \qquad B = \det X\,, \tag{3.14}$$

in case (b), we have

$$\begin{array}{l}\dim V_1 = N \\ \dim V_2 = N\end{array} \qquad \dim V_3 = 2N \qquad B = \det \begin{pmatrix} X & Y \end{pmatrix}\,, \tag{3.15}$$

for (c), the dibaryon satisfies

$$\begin{array}{l}\dim V_1 = 3N \\ \dim V_2 = 3N\end{array} \qquad \begin{array}{l}\dim V_3 = 2N \\ \dim V_4 = 2N \\ \dim V_5 = 2N\end{array} \qquad B = \det \begin{pmatrix} X & 0 \\ Y & Y' \\ 0 & Z \end{pmatrix}\,, \tag{3.16}$$

and, finally, for (d),

$$\begin{array}{l}\dim V_1 = N \\ \dim V_2 = N\end{array} \qquad \dim V_3 = 2N \qquad B = \det \begin{pmatrix} YX & Z \end{pmatrix}\,. \tag{3.17}$$

One advantage of this condensed notation in terms of determinants — instead of the explicit notation in terms of ϵ-symbols — is that it is quite easy to see which paths give factorized dibaryons, or which paths do not actually yield any dibaryon at all because the

then define the determinant of the map as the determinant of its matrix representation with respect to some unitary basis. This definition is independent of the choice of unitary basis. We thank Andrei Mikhailov for raising this question.

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determinant is zero. A zero determinant can occur, for example, if the multiplicities of the vector spaces is larger than the number of arrows between them, so that the same arrow must appear several times. Moreover, it is also quite easy to formulate deformations of dibaryons. For instance, if between V_t and V_h we have two maps A_1 and A_2, we can consider the determinant of the general linear combination

$$B(\lambda_1, \lambda_2) = \det(\lambda_1 A_1 + \lambda_2 A_2)\,. \tag{3.18}$$

Note that we must require $|\lambda_1|^2 + |\lambda_2|^2 = 1$ for the dibaryon to be properly normalized and that we identify modulo the phase $(\lambda_1, \lambda_2) \equiv e^{i\phi}(\lambda_1, \lambda_2)$. In other words, the moduli of these dibaryons is the projective space $\mathbb{P}^1$. The existence of such a moduli space is also a typical situation on the geometry side. In quantizing this moduli space, as explained in [12], the coupling to the RR flux leads to the identification of the ground state Hilbert space with the space of sections of $\mathcal{O}_{\mathbb{P}^1}(N)$, which has dimension $N+1$. In field theory, these ground states correspond to the homogeneous terms in the expansion of $B(\lambda_1, \lambda_2)$ in powers of (λ_1, λ_2), and there are also exactly $N+1$ of them.

4. The geometry of dibaryons

In AdS/CFT, dibaryons correspond to D3-branes wrapped on supersymmetric 3-cycles in $\mathbf{Y}$ which project to holomorphic curves in $\mathbf{V}$. On the gauge theory side, because they are constructed as antisymmetrized products of bifundamental chiral fields, the dibaryons of our quiver theories carry charges $Q_I(B)$ under the baryonic U(1) symmetries discussed in subsection 3.1. Moreover, they also carry a definite R-charge $R(B)$. Geometrically, the U(1) charges are naturally identified with the homology class of the corresponding curves.

We review briefly a construction from [16, 46, 47] in order to better understand the stability of these D3-brane wrappings. We start with a euclidean D3-brane wrapping a holomorphic, four-dimensional cycle in the Calabi-Yau cone $\mathbf{X}$. The total space is $\mathbb{R}^4 \times \mathbf{X}$. Now we move to a $H_5 \times \mathbf{Y}$ geometry by adding a large amount of five-form flux. The space $\mathbf{Y}$ is five dimensional and is Sasaki-Einstein. The space H_5 is euclidean AdS_5 and is highly symmetric. In particular, we can choose the old radial direction of $\mathbf{X}$ to Wick rotate. After Wick rotation, H_5 will turn into AdS_5. The lowest energy dibaryons are typically time independent. Tracing back through the above construction, we see that time independent wrappings corresponded initially to D3-brane wrappings that were independent of the radial coordinate. Moreover, the wrappings were holomorphic and the radius is paired holomorphically with the U(1) coordinate in the Sasaki-Einstein space $\mathbf{Y}$. Thus these wrappings are also independent of the U(1) angle. In other words, time independent dibaryons wrap holomorphic curves $C \subset \mathbf{V}$ along with the U(1) fiber above every point in C.

In the geometric identification of the global U(1) symmetries, the simplest charge to understand is the R-charge. Roughly speaking one expects the mass of the D3-brane to be proportional to the tension times the volume of this wrapped three-cycle in $\mathbf{Y}$. Moreover, for objects in AdS/CFT correspondence with mass of order N, the mass and conformal dimension are equal up to corrections of order $1/N$. Finally, dibaryons are chiral primary

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operators so we expect that their conformal dimension $\Delta = 3R/2$, where R is the R-charge. On the other hand, because $\mathbf{V}$ is Einstein, the volume of the curve C is proportional to its intersection with the canonical class $K_{\mathbf{V}}$ of $\mathbf{V}$, i.e., to its degree $\deg(C) = -K_{\mathbf{V}} \cdot C$. Taken together, the following exact result of [16, 17] should not be surprising,

$$R(B) = \frac{2N}{K_{\mathbf{V}}^2} \deg(C) . \tag{4.1}$$

The constant of proportionality comes from carefully considering the D3-brane tension and the metric on $AdS_5 \times \mathbf{Y}$. The fact that the R-charge, and not the mass shows up on the left hand side of the equation was discussed by [48], and comes from the existence of fermionic zero modes in AdS_5 for these dibaryons.

Beyond the R-charge, one would like also to identify all other baryonic U(1) charges Q_I unambiguously with geometric quantities. As discussed in section 3, these other U(1)'s are naturally associated to the remaining generators of $H^2(\mathbf{V})$, but it is not yet clear which U(1)'s correspond to which elements of the homology class. We fix this ambiguity in section 4.3 by identifying curves C with antisymmetric products of bifundamental fields. We also discuss a number of consequences and then provide extensive tests for $\mathbf{V}$ a del Pezzo surface. First, however, we discuss the intersection form on the curves C in greater detail.

4.1 Charges and intersection form

One interesting property of curves in surfaces is that they intersect in points, and it is natural to ask what could be the analog of the intersection product on the gauge theory side. In search for such a quantity, let us rewrite the maximizing-a principle (3.9) as the condition

$$\operatorname{tr} R^2 Q_I = 0 \qquad \text{for all } I, \tag{4.2}$$

which ensures that we are at a critical point of a, together with

$$\mathfrak{K} < 0 , \qquad \text{where } \mathfrak{K} \text{ is the matrix } \mathfrak{K}_{IJ} = \operatorname{tr} R Q_I Q_J , \tag{4.3}$$

which ensures that we have a maximum. In fact, the maximizing-a principle was derived from precisely these two conditions in [36]. The matrix $\mathfrak{K}_{IJ}$ is symmetric because the Q_I commute with R, and it is tempting to interpret $\mathfrak{K}$ as an intersection form. We will see that this identification is indeed correct, at least if $\mathbf{V}$ is a del Pezzo surface. More precisely, we want to view $\operatorname{tr} RQ_1Q_2$ as the intersection of the two charges Q_1 and Q_2, which can be the generators of the U(1) R-symmetry or of any of the baryonic U(1)'s. Then (4.2) says that the R-charge is orthogonal to the baryonic U(1)'s while (4.3) is the statement that the orthogonal complement of the R-charge has a definite signature. On the del Pezzo side, the corresponding statement is that the orthogonal complement of the canonical divisor K is negative definite, and is in fact isomorphic to the root lattice of the exceptional simply laced Lie algebras, see, e.g., [49]. Via AdS/CFT, the canonical divisor corresponds to the R-charge and the orthogonal complement to the baryonic U(1)'s.

To make this explicit, let us denote by $C(B)$ the holomorphic curve that corresponds to the dibaryon B. The projection orthogonal to the canonical divisor K is given by

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$C^\perp = C - K(K \cdot C)/K^2$. The formula that we will compare between gauge theory and geometry is then explicitly

$$C^\perp(B_1) \cdot C^\perp(B_2) = \frac{1}{2} Q_I(B_1) \mathfrak{K}^{IJ} Q_J(B_2)\,, \tag{4.4}$$

where the l.h.s. is the geometric intersection product in the surface $\mathbf{V}$, while the r.h.s. is a gauge theory expression in terms of the U(1) charges of two dibaryons and the inverse $\mathfrak{K}^{IJ}$ of the matrix $\mathfrak{K}_{IJ}$.[7]

4.2 Dual exceptional collections

We recall that the gauge theory of interest was obtained starting from an exceptional collection of bundles $\mathcal{E} = (E_1, E_2, \ldots, E_m)$, where $m = n+3$ is $\dim K_0(\mathbf{V})$ of the n-th del Pezzo surface $\mathbf{V} = dP_n$. To each E_i corresponds a node in the quiver and to each $\mathrm{Ext}^k(E_i, E_j)$ a chiral field X_{ij}. A dibaryon B is constructed by antisymmetrization from a certain combination of the X_{ij}. Our basic claim is that one can read off the class of the curve corresponding to B using not the exceptional collection $\mathcal{E}$, but a certain dual exceptional collection, $\mathcal{E}^\vee$, which we define presently. It is important to point out that the original collection $\mathcal{E}$ lives on the large volume Calabi-Yau (and, by continuity, on the singular cone $\mathbf{X}$), while the dual collection $\mathcal{E}^\vee$ lives on the large-N dual geometry $\mathbf{Y} \to \mathbf{V}$.

For any exceptional collection $\mathcal{E} = (E_1, E_2, \ldots E_m)$, we define a dual collection $\mathcal{E}^\vee$ as the result of a certain braiding operation,

$$\begin{aligned} \mathcal{E}^\vee &= (E_m^\vee, E_{m-1}^\vee, \ldots, E_1^\vee) \\ &= (L^D_{E_1} \cdots L^D_{E_{m-1}} E_m, L^D_{E_1} \cdots L^D_{E_{m-2}} E_{m-1}, \ldots, L^D_{E_1} E_2, E_1)\,. \end{aligned} \tag{4.5}$$

The collection $\mathcal{E}^\vee$ is exceptional in the order presented, and is dual to $\mathcal{E}$ in the sense of the Euler form, i.e., $\chi(E_i, E_j^\vee) = \delta_{ij}$. This property of the collection (4.5) is easily checked based on the transformation of the charges under braiding, eq. (2.11), linearity of χ, and the upper triangularity of the matrix $S_{ij} = \chi(E_i, E_j)$.

We note that dual exceptional collections play an important role also in the weak coupling limit in the context of the McKay correspondence [25, 26], and in mirror symmetry [30]. The role they play in AdS/CFT is new, but has been hinted at in [17, 51].

[7] This identification of $\mathrm{tr}\, RQ_IQ_J$ as an intersection form is reminiscent of a similar formula on the string worldsheet. In the context of D-branes on Calabi-Yau spaces (in fact, also in flat space), it is well-known that the intersection of two D-branes D_1 and D_2 can be computed from the Witten index in the Ramond sector of open strings stretched between D_1 and D_2,

$$D_1 \cdot D_2 = \mathrm{tr}_{\mathcal{H}^{\mathrm{R}}_{D_1 D_2}} (-1)^F\,, \tag{$*$}$$

see, e.g., [30, 50]. Upon modular transformation to the closed string sector, the expression ($*$) becomes the overlap of the corresponding boundary states

$$D_1 \cdot D_2 = \langle D_1 | e^{-\pi i J_0} e^{-2\pi\tau H_{\mathrm{cl}})} | D_2 \rangle_{\mathrm{RR}} \tag{$**$}$$

in the Ramond-Ramond sector with insertion of the R-charge, with only ground states contributing in the limit of $\tau \to \infty$. Our formula (4.4) can be viewed as the holographic version of eq. ($**$). It would be interesting to see whether there is also a gauge theory expression for ($*$).

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One interesting property of the dual collection is that the ranks of the dual bundles are proportional to the ranks Nd_i of the gauge groups in the original quiver. This follows from the fact that the charge of the D3 brane, which is the class of a point in $\mathbf{V}$, decomposes as[8]

$$\sum_i d_i \,\mathrm{ch}(E_i) = \mathrm{ch}(\mathcal{O}_p)\,, \tag{4.6}$$

and multiplying with $\chi(\cdot, E_i^\vee)$ yields the equality $d_i = r(E_i^\vee)$. In particular, since all $d_i > 0$, all $E_i^\vee$ are actual bundles (and not anti-bundles), as befits a large volume description. Moreover, multiplying (4.6) with $\chi(\mathcal{O}_p, \cdot)$ yields

$$\sum_i r(E_i^\vee) r(E_i) = \sum_i \chi(\mathcal{O}_p, E_i^\vee)\chi(E_i, \mathcal{O}_p) = \chi(\mathcal{O}_p, \mathcal{O}_p) = 0\,. \tag{4.7}$$

We can also investigate the χ-dual of a sequence of mutations on $\mathcal{E} = (E_1, E_2, \ldots, E_n)$. Using (2.10), it is easy to see that

$$(L_i\mathcal{E})^\vee = L_{n-i}\mathcal{E}^\vee\,, \tag{4.8}$$

which implies that for $j > i$

$$(L_{j-1}L_{j-2}\cdots L_{i+1}L_i\mathcal{E})^\vee = L_{n-j+1}\cdots L_{n-i-1}L_{n-i}(\mathcal{E}^\vee)\,, \tag{4.9}$$

This relation is interesting because in some cases the sequence of mutations on the l.h.s. of (4.9) is equivalent to Seiberg duality in the gauge theory. The r.h.s. of (4.9) then gives a straightforward way to determine the ranks of the gauge groups and charges of the fields after Seiberg duality.

4.3 Dibaryon charges from exceptional collections

To identify the charges of the dibaryons in geometry, it will suffice to specify the classes of the generating fields X_{ij}. We note that the classes that we associate with the X_{ij} will in general be fractional, and that only the combinations of X_{ij} that upon antisymmetrization give non-trivial dibaryons on the field theory side will correspond to integral classes with actual curves as representatives.

Consider an arrow $a = (ij)$ connecting two nodes in the quiver, with corresponding bifundamental field X_{ij}. The sheaf that is dual to the node at the tail is $E_{t(a)}^\vee = E_i^\vee$, and the sheaf that is dual to the head is $E_{h(a)}^\vee = E_j^\vee$. We distinguish two cases. Either $\chi(E_i^\vee, E_j^\vee) \neq 0$ or $\chi(E_i^\vee, E_j^\vee) = 0$. In the first case, the fractional divisor is

$$D_a = D_{ij} = \frac{c_1(E_j^\vee)}{r(E_j^\vee)} - \frac{c_1(E_i^\vee)}{r(E_i^\vee)}\,. \tag{4.10}$$

In the second case, we add $-K$ to the divisor: $D_a \to D_a - K$. In all cases we have studied so far, the resulting divisor has positive degree, $D_a \cdot (-K) > 0$. This is reassuring, since we know that the R-charge in field theory is related to the degree by $R(X_a) = 2\deg(D_a)/K^2$, and we want the R-charges of all chiral fields to be positive.

[8] From now on, $d_i \equiv d^i$.

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To see how the formula (4.10) associates integral classes with dibaryons, recall that on the field theory side we need to take enough and appropriate kinds of bifundamental matter fields to antisymmetrize completely over the color indices. As explained in subsection 3.2, the condition is that the dimension of the vector spaces at head and tail be equal. Since $d_i = r(E_i^\vee)$, the condition that there is no uncontracted fundamental index is that the differences of ranks in the dual collection be zero, i.e., that there is no D5-brane charge. In the simplest case where we antisymmetrize over only one bifundamental matter field, the dibaryon contains $N \operatorname{lcm}(d_i, d_j)$ copies of the field (which are distinguished by their flavor index). Recalling from the previous subsection that $d_i = r(E_i^\vee)$, we see that the factor $\operatorname{lcm}(d_i, d_j)$ is of just the right form to place the curve C corresponding to the dibaryon in the integer homology

$$C = \frac{r(E_i^\vee)c_1(E_j^\vee) - r(E_j^\vee)c_1(E_i^\vee)}{\gcd(r(E_i^\vee), r(E_j^\vee))} . \tag{4.11}$$

This example captures the spirit of the process of constructing dibaryons, but is an oversimplification. Even though the coefficients in C may be integers, the divisor may still not be good for wrapping branes. Recall that we may compute the genus of a curve from the adjunction formula $2g-2 = C\cdot(C+K)$. Many times, an arbitrary C, even if integral, would have $g < 0$! In practice, we have often found that when $g < 0$, there is some corresponding gauge theory obstruction which makes the corresponding antisymmetrization vanish, for example because w_a is too small. We do not, however, know a general proof why the dibaryon should vanish whenever there is no corresponding curve to wrap.

Using (4.10), we can give a geometric and simple formula for the R-charge of the bifundamental matter fields, obtained by intersecting D_{ij} with $-K$,

$$R(X_{ij}) = \frac{2}{K^2 r(E_i^\vee) r(E_j^\vee)} \times \begin{cases} \chi(E_i^\vee, E_j^\vee) & \text{if } \chi(E_i^\vee, E_j^\vee) \neq 0 \\ \chi(E_i^\vee \otimes K, E_j^\vee) & \text{otherwise.} \end{cases} \tag{4.12}$$

where $E_i^\vee$ and $E_j^\vee$ live in the dual collection. The constraint that the R-charge of a chiral field be positive becomes the condition that $\chi(E_i^\vee, E_j^\vee) \geq 0$ and $\chi(E_j^\vee, E_i^\vee) < r(E_j^\vee) r(E_i^\vee) K^2$.

This R-charge formula (4.12) has a remarkable property under reversal in arrow direction. If we denote by X_{ji} the antichiral field conjugate to X_{ij}, then (4.12) yields $R(X_{ij}) = 2 - R(X_{ji})$, as one can easily check using (2.4). Recall that the fermions in the chiral multiplet have R-charge $R_f = R - 1$. Thus, under this switch in arrow direction, $R_f \to -R_f$, as one would have expected.

For the reader interested where this identification between bifundamentals and fractional divisors comes from, it builds on work of [17] and was arrived at by studying a large number of examples, some of which we will describe in the following sections. However, having intuited the relationship between the X_{ij} and the lattice of divisors, we can now go back and make sure these formulae have the right properties.

First, we can check that loops in the quiver, which can correspond to terms in the superpotential, can have R-charge two. The R-charge of such a loop is proportional to the

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degree of the sum of the fractional divisors D_a over all arrows a in that loop,

$$R_{\text{loop}} = \frac{2}{K^2} \sum_{a \in \text{loop}} d(D_a) \,. \tag{4.13}$$

Moreover, from the definition of the fractional divisors, the sum over the D_a must be an integer multiple of $c_1(\mathbf{V}) = -K$ (all the other classes cancel in the sum). Therefore,

$$R_{\text{loop}} = 2n \,. \tag{4.14}$$

Finally, from the structure of the exceptional collection, $n \geq 1$. We conclude that the loop can appear in the superpotential only if $n = 1$.

Second, we can test that (4.12) satisfies the NSVZ beta functions. The NSVZ beta functions (3.3) can be written for each node i as

$$\beta_i = 2d_i^2 + \sum_j R_f(X_{ij}) d_i d_j (\chi(E_i, E_j) - \chi(E_j, E_i)) \,. \tag{4.15}$$

After some manipulation, this expression reduces to

$$\beta_i = \frac{2}{K^2} \sum_k (\chi(E_i^\vee, E_k^\vee) - \chi(E_k^\vee, E_i^\vee))(\chi(E_i, E_k) - \chi(E_k, E_i)) + 2r(E_i^\vee) r(E_i) \,. \tag{4.16}$$

To show that the β_i do indeed vanish, we will show something stronger, namely that the matrix $NSVZ_{ij}$ vanishes identically, where

$$NSVZ_{ij} = \frac{2}{K^2} \sum_k (\chi(E_i^\vee, E_k^\vee) - \chi(E_k^\vee, E_i^\vee))(\chi(E_j, E_k) - \chi(E_k, E_j)) + 2r(E_i^\vee) r(E_j) \,. \tag{4.17}$$

The proof can be described easily and falls into two pieces. First, one shows that under the basis transformation $F_j = B_{ij} E_i$,

$$NSVZ \to B^{-1} \cdot (NSVZ) \cdot B \,. \tag{4.18}$$

Second, one shows that $NSVZ$ vanishes in a particular basis. A convenient choice is the basis in which the sheaves are expressed in terms of their rank, first Chern class, and second Chern character, i.e. the basis in which χ is written as in (2.4). In this basis $2r(E_i^\vee) r(E_j)$ is a matrix which is zero everywhere except for a two in the lower left hand corner. The first term in (4.17) is similarly a matrix which is zero everywhere except for a -2 in the lower left hand corner.

In the same way that $R(X_{ij})$ was obtained by intersecting the fractional divisor D_{ij} with K, the $Q_I(X_{ij})$ are obtained by intersecting the D_{ij} with the generators of the lattice orthogonal to K. It would be ideal to finish this section with a demonstration that this choice of R_{ij} maximizes the conformal anomaly a [36] over the space of Q_I for a general quiver and corresponding exceptional collection, but we have as yet only been able to prove it for three block exceptional collections and on a case by case basis.

We now turn to the examples.

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5. Simple examples of exceptional collections

Having tackled the surface $\mathbb{P}^2$ in the introduction, we consider dibaryons arising from holomorphic curves in the slightly more complicated surfaces $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ blown up at one or two points.

5.1 The first del Pezzo

There are in fact two del Pezzos with $K^2 = 8$. One is $\mathbb{P}^1 \times \mathbb{P}^1$ and the other is $\mathbb{P}^2$ blown up at a point. The sheaves on both surfaces are easy to describe. The second Betti number of both surfaces $b_2 = 2$ and so we need two weights to describe the lattice of divisors.

We begin with $\mathbb{P}^1 \times \mathbb{P}^1$. A divisor can be written $D = mf + ng$ where $f \cdot f = 0$, $g \cdot g = 0$, $f \cdot g = 1$, and m, $n \in \mathbb{Z}$. The canonical divisor $K = -2f - 2g$. Exceptional collections on $\mathbb{P}^1 \times \mathbb{P}^1$ are well known. For example, [22] gives the collection $(\mathcal{O}(-f-g), \overline{\mathcal{O}(-f)}, \overline{\mathcal{O}(-2g)}, \mathcal{O}(-g))$. In this basis, S_{ij} is indeed upper triangular

$$S = \begin{pmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.1}$$

The dual collection is $\mathcal{E}^\vee = (\mathcal{O}(-2f-3g), \mathcal{O}(-2f-2g), \mathcal{O}(-f-2g), \mathcal{O}(-f-g))$ and the corresponding $(S^{-1})_{ij}$ is

$$S^{-1} = \begin{pmatrix} 1 & 2 & 4 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.2}$$

From the ranks of the dual bundles, we infer that the ranks of the gauge groups in the quiver are all SU(N).

We can also read off the R-charges of the generators of the algebra. From (4.12), we see that the R-charges of the generators are all $1/2$. This charge agrees with the possible dibaryon R-charges which are integer multiples of $N/2$.

We can also compute the "fractional" divisors, which are in fact not fractional in this example, for each bifundamental field. These divisors are nothing but f and g. Antisymmetrizing over the bifundamental fields corresponds to constructing the holomorphic curve $C = af + bg$ where a and b are non-negative integers.

The next example is $\mathbb{P}^2$ blown up at a point. We denote by H the hyperplane of $\mathbb{P}^2$ and by E the exceptional divisor. The most general divisor can then be written $D = aH + bE$ where a, $b \in \mathbb{Z}$ and $H \cdot H = 1$, $E \cdot E = -1$, and $H \cdot E = 0$. Exceptional collections are well known for this example as well. We take $\mathcal{E} = (\mathcal{O}, \overline{\mathcal{O}(H-E)}, \overline{\mathcal{O}(E)}, \mathcal{O}(H))$ from [22]. In this basis

$$S = \begin{pmatrix} 1 & -2 & -1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.3}$$

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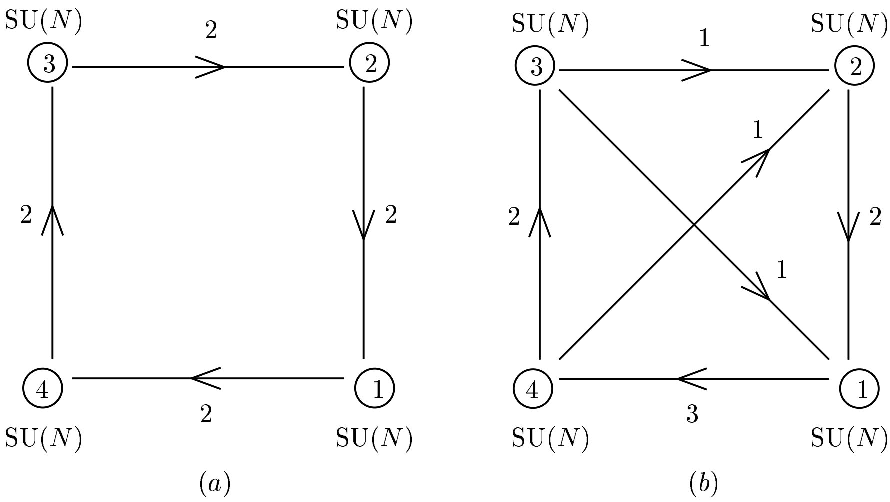

Figure 3: Quivers for (a) $\mathbb{P}^1 \times \mathbb{P}^1$ and (b) $\mathbb{P}^2$ blown up at a point.

The dual collection is $\mathcal{E}^\vee = (\mathcal{O}(-2H+E), \mathcal{O}(-H), \mathcal{O}(-H+E), \mathcal{O})$ and

$$S^{-1} = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{5.4}$$

The ranks of the gauge groups are again all $\mathrm{SU}(N)$ for this quiver. Moreover, we can read off the R-charges and fractional divisors of the generators

	D	R
X_{32}	E	1/4
X_{21}	$H-E$	1/2
X_{43}	$H-E$	1/2
X_{42}	H	3/4
X_{31}	H	3/4
X_{14}	H	3/4

(5.5)

These same R-charges can be calculated by maximizing the conformal anomaly a [36]. Moreover, this table agrees with the table (4.13) of [17] where the authors arrived at the D in a rather different way.

5.2 The second del Pezzo

A divisor for the second del Pezzo ($\mathbb{P}^2$ blown up by two exceptional divisors E_1 and E_2) may be written $D = aH + b_1E_1 + b_2E_2$ where a and b_i are integers and $H \cdot H = 1$, $E_i \cdot E_j = -\delta_{ij}$, and $H \cdot E_i = 0$.

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Exceptional collections are well known for the second del Pezzo. One such collection is $\mathcal{E} = (\mathcal{O}, \overline{\mathcal{O}(H)}, \overline{\mathcal{O}(2H - E_1 - E_2)}, \mathcal{O}(2H - E_1), \mathcal{O}(2H - E_2))$. In this basis

$$S = \begin{pmatrix} 1 & -3 & -4 & 5 & 5 \\ 0 & 1 & 1 & -2 & -2 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.6}$$

We show the quiver corresponding to $\mathcal{E}$ in figure 4. The inverse matrix is

$$S^{-1} = \begin{pmatrix} 1 & 3 & 1 & 2 & 2 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.7}$$

This example is the first we have come across so far that requires the use of higher rank bundles. In particular, the dual collection is $\mathcal{E}^\vee = (\mathcal{O}(-H + E_1), \mathcal{O}(-H + E_2), \mathcal{O}(-H + E_1 + E_2), F, \mathcal{O})$ where the charges of F are $\mathrm{ch}(F) = (2, -H, -1/2)$. Because the rank of F is two, there will now be one gauge group in the quiver, corresponding to $\overline{\mathcal{O}(H)} \in \mathcal{E}$, with gauge group $\mathrm{SU}(2N)$ instead of $\mathrm{SU}(N)$.

Using the formulae for the R-charges and the fractional divisors, we find that

	D	R
X_{42}	$\frac{1}{2}H - E_2$	$1/7$
X_{52}	$\frac{1}{2}H - E_1$	$1/7$
X_{31}	$H - E_1 - E_2$	$2/7$
X_{43}	E_1	$2/7$
X_{53}	E_2	$2/7$
X_{21}	$\frac{1}{2}H$	$3/7$
X_{14}	$2H - E_1$	$10/7$
X_{15}	$2H - E_2$	$10/7$
X_{23}	$\frac{5}{2}H$	$15/7$

(5.8)

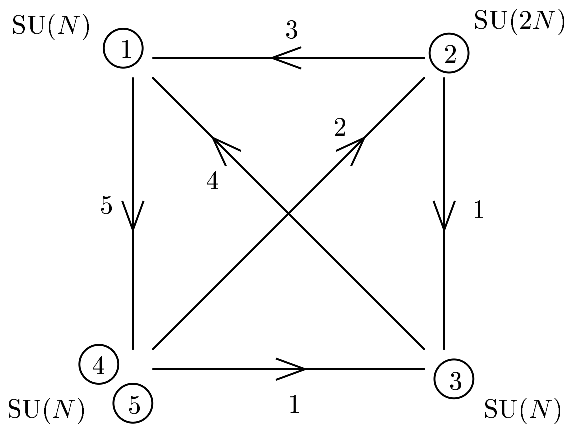

Figure 4: Quiver for the second del Pezzo.

These same R-charges can be obtained by maximizing the conformal anomaly a with some weak assumptions about the form of the superpotential.

It is fun to see how these fractional divisors combine to give a dibaryon. From gauge theory, there is a dibaryon that can be constructed from an antisymmetric product of N X_{42} fields and N X_{52} fields. From the geometric perspective, we see that the fractional divisors for these fields combine to give the curve $H - E_1 - E_2$, a degree one curve corresponding to one of the smallest wrapped D3-branes in the dibaryon spectrum.

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We can also check our formula (4.4) for the intersection of the dibaryons. In the Q_a basis for the charges, where we take the ordering of arrows to be as in (5.8), the two baryonic U(1) charges satisfying (3.5) are given by $Q_1 = (-3,-1,-4,2,4,1,2,0,5)$ and $Q_2 = (1,-1,0,1,-1,0,-1,1,0)$. Using the R-charges in (5.8), one then finds

$$\mathfrak{K} = \operatorname{tr} RQ_IQ_J = \begin{pmatrix} -32 & 4 \\ 4 & -4 \end{pmatrix} . \tag{5.9}$$

The charges Q_1 and Q_2 correspond geometrically to the intersection with the divisors orthogonal to the canonical class $-3H + E_1 + E_2$. It is easy to check that

$$\begin{aligned} &2H - 4E_2 - 2E_1 \, , \\ &E_2 - E_1 \end{aligned} \tag{5.10}$$

indeed have the same mutual intersection products as in (5.9) (up to a factor of 2), and that the intersection with the fractional divisors in (5.8) are the same as the charges of the corresponding chiral fields.

Let us note that in principle, one could have tried to derive the identifications in (5.8) just by looking at the charges of the fields and their intersection product in field theory (assuming one knew about the significance of $\operatorname{tr} RQ_IQ_J$). However, the identification of this data with the geometry is ambiguous because of the discrete symmetries shared by the field theory and the del Pezzo. This symmetry is a $\mathbb{Z}_2$ in the case of the second del Pezzo and exchanges nodes 4 and 5 in the quiver, or E_1 and E_2 in the geometry. In general, the n-th del Pezzo carries the action of the Weyl group of E_n and the quiver theory has the same symmetries. Because the intersection form is invariant under the Weyl group, one is left with a Weyl group ambiguity in identifying charges with divisors. Our prescription using the dual exceptional collection fixes this ambiguity, and is only invariant under the simultaneous action on field theory and geometry.

6. Three block collections, anomalies, and Markov

Three block exceptional collections provide some of the most impressive evidence in support of our prescription for identifying bifundamental fields X_{ij} with fractional divisors D_{ij} in the dual geometry. Our plan in this section is to present first a gauge theory computation or description and then to relate the gauge theory to a property of the exceptional collection.

Consider a quiver consisting of three blocks of nodes (see figure 5). The nodes within a block are not joined by arrows. The gauge groups of the nodes in a block are all the same. Between any two representative nodes in two distinct blocks, there are the same number of arrows. Let there be α nodes in the first block, β nodes in the second block, and γ nodes in the third block. Let the ranks of the gauge groups be xN, yN, and zN. Let there be a arrows between nodes in the second and third blocks, b arrows between nodes in the third and first blocks, and c arrows between nodes in the first and second block.

These quivers derive from three block exceptional collections $(\mathcal{E}, \mathcal{F}, \mathcal{G})$. For any two sheaves within a block E_i, $E_j \in \mathcal{E}$, $\chi(E_i, E_j) = 0$. Moreover, for two sheaves in different

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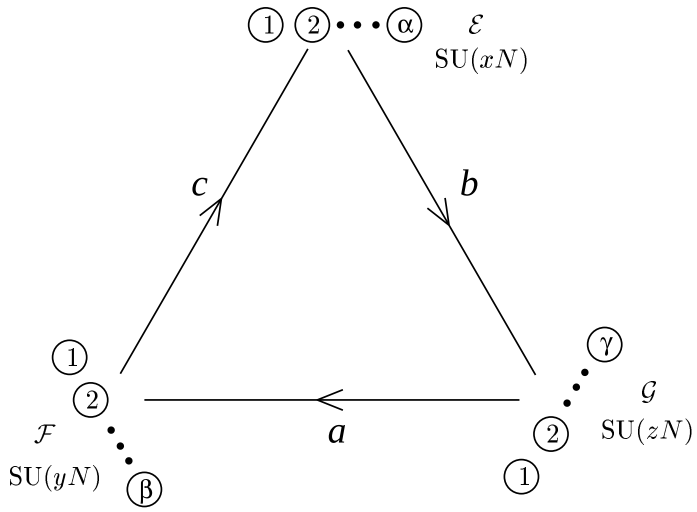

Figure 5: Quiver for a three-block exceptional collection.

blocks $E_i \in \mathcal{E}$ and $F_j \in \mathcal{F}$, $\chi(E_i, F_j)$ is independent of i and j. These three block collections satisfy in addition all the properties of ordinary exceptional collections, and were discussed in great detail in [33, 34]. Three block exceptional collections exist for $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and all dP_n for $n > 2$.

To see that the ranks of the gauge groups within a block are all the same from $(\mathcal{E}, \mathcal{F}, \mathcal{G})$ requires more effort. The requirement $\chi(E_i, E_j) = 0$ means that $r(E_i)/\deg(E_i)$ is independent of i. One can then argue on more general grounds that the rank and degree are coprime [42]. The statement about the gauge groups follows from the fact that the χ-dual of $(\mathcal{E}, \mathcal{F}, \mathcal{G})$ is also a three block exceptional collection, $(\mathcal{G} \otimes K, L^D_{\mathcal{E}}\mathcal{F}, \mathcal{E})$.

We now analyze the gauge theory for these three block quivers. The cancellation of the chiral anomaly requires, roughly speaking, that the number of arrows into a node equals the number of arrows out from that node. More precisely,

$$a\beta y = b\alpha x \, ; \qquad b\gamma z = c\beta y \, ; \qquad c\alpha x = a\gamma z \, . \tag{6.1}$$

These relations allow us to set a, b and c in terms of the other variables up to a constant of proportionality K'. In particular, we choose

$$a = \alpha K' x \, ; \qquad b = \beta K' y \, ; \qquad c = \gamma K' z \, . \tag{6.2}$$

Knowing where this calculation is headed, we choose

$$K' \equiv \sqrt{\frac{K^2}{\alpha\beta\gamma}} \, . \tag{6.3}$$

We want our gauge theory to be conformal, and thus we need the NSVZ beta functions to vanish for each node:

$$x + \frac{1}{2}(b\gamma z(R_b - 1) + c\beta y(R_c - 1)) \, = 0 \, , \tag{6.4}$$

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$$y + \frac{1}{2}(c\alpha x(R_c - 1) + a\gamma z(R_a - 1)) = 0\,, \tag{6.5}$$
$$z + \frac{1}{2}(a\beta y(R_a - 1) + b\alpha x(R_b - 1)) = 0\,, \tag{6.6}$$

where R_a, R_b, and R_c are the R-charges of the bifundamental matter fields.

We also insist that the superpotential, which we assume to be cubic, have R-charge two. This constraint $R_a + R_b + R_c = 2$ implies the following Markov type equation for the quiver

$$\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} = abc\,. \tag{6.7}$$

We can rewrite this equation in terms of x, y, and z,

$$\alpha x^2 + \beta y^2 + \gamma z^2 = \sqrt{K^2\alpha\beta\gamma}xyz\,. \tag{6.8}$$

This Markov equation featured prominently in [34] where it was used to classify all three block exceptional collections. Geometrically, this equation can be understood as an invariant of the matrix $S_{ij} = \chi(E_i, E_j)$. The trace $\operatorname{tr} S^{-1}S^t$ is invariant under change of basis. In the special basis where the sheaves are written in terms of their rank, c_1, and ch_2, it is easy to check that $\operatorname{tr} S^{-1}S^t = 12 - K^2 = c_2(\mathbf{V})$. In the basis where the sheaves form a three block exceptional collection, S takes the form

$$S = \begin{pmatrix} \mathrm{Id}_\alpha & -C & B \\ 0 & \mathrm{Id}_\beta & -A \\ 0 & 0 & \mathrm{Id}_\gamma \end{pmatrix} \tag{6.9}$$

where Id_n is an $n \times n$ identity matrix. Also, A, B, and C are rectangular matrices where every entry is respectively a, b, or c. In order for $\operatorname{tr} S^{-1}S^t$ to remain invariant, the Markov equation (6.7) must hold.

Continuing with the gauge theory analysis, we find simple expressions for the R-charges

$$R_a = \frac{2a}{\alpha bc}\,; \qquad R_b = \frac{2b}{\beta ac}\,; \qquad R_c = \frac{2c}{\gamma ab}\,. \tag{6.10}$$

These values agree with the geometric result (4.12).

We now consider the R-charges of dibaryonic operators in the gauge theory. Assume such an operator is constructed from mN a-type bifundamentals, nN b-type bifundamentals, and pN c-type bifundamentals. The total R-charge is thus

$$R/N = mR_a + nR_b + pR_c\,. \tag{6.11}$$

Using the expressions (6.10), we can rewrite the total R-charge as

$$R/N = \frac{2}{K^2}\frac{1}{xyz}(max + nby + pcz)\,. \tag{6.12}$$

The dibaryons are constructed from antisymmetrizing over the fundamental indices of the matter fields. To be able to antisymmetrize, we need $x|(n-p)$, $y|(m-p)$, and $z|(m-n)$.

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We also take the x, y, and z to be relatively prime. These conditions, along with the fact (which is a simple consequence of (6.8)) that

$$ax + by + cz \equiv 0 \mod xyz \tag{6.13}$$

imply that the total R-charge of the dibaryon is an integer multiple of $2/K^2$ in agreement with the geometric prediction (4.1).

We move on to a calculation of the cubic anomalies for these theories. First we consider the $\operatorname{tr} R^3$ anomaly.

$$\begin{aligned}\frac{1}{N^2}\operatorname{tr} R^3 &= \alpha x^2 + \beta y^2 + \gamma z^2 + \\ &\quad + \beta\gamma yza(R_a - 1)^3 + \alpha\gamma xzb(R_b - 1)^3 + \alpha\beta xyc(R_c - 1)^3 \,.\end{aligned} \tag{6.14}$$

Using (6.8), this sum can be reduced to

$$\frac{1}{N^2}\operatorname{tr} R^3 = \frac{24}{K^2}\,. \tag{6.15}$$

This result agrees with an independent prediction for the conformal anomaly a_c in terms of the volume of the Sasaki-Einstein manifold $\mathbf{Y}$.[9] From (3.9), and the fact that $\operatorname{tr} R = 0$, we know that $\operatorname{tr} R^3 = 32a_c/9$. The vanishing of $\operatorname{tr} R$ also means that $a_c = c_c$ where c_c is the other central charge in the superconformal algebra [44, 45]. The value of c_c for $\mathbf{S}^5$ was calculated long ago by [2] to be $N^2/4$ and is easy to generalize to arbitrary $\mathbf{Y}$ (knowing that c_c is proportional to $1/\operatorname{Vol}(\mathbf{Y})$ [52, 53]):

$$a_c = c_c = \frac{N^2}{4}\frac{\operatorname{Vol}(\mathbf{S}^5)}{\operatorname{Vol}(\mathbf{Y})}\,. \tag{6.16}$$

For the del Pezzos,

$$\operatorname{Vol}(\mathbf{Y}) = \frac{\pi^3}{27}K^2 \tag{6.17}$$

as can be seen from, for example, [54]. Putting (6.16), (6.17), and (3.9) together, we get precisely (6.15).

There are also flavor U(1) symmetries. It is easy to describe the baryonic U(1)'s discussed in subsections 3.1 and 4.1. We order the nodes in each block in a line and label them by their block and position. We find a baryonic U(1) for each pair of adjacent nodes in a block as follows. An ingoing bifundamental on the first node and an outgoing bifundamental on the second node will have charge 1 under this U(1). An outgoing bifundamental on the second node and an ingoing bifundamental on the first node will have charge -1 under the U(1). We label the symmetry currents associated with these U(1)'s as Q_I, where I runs over all pairs. There are $\alpha - 1$ such pairs from $\mathcal{E}$, $\beta - 1$ from $\mathcal{F}$ and $\gamma - 1$ from $\mathcal{G}$, for a total of n baryonic U(1)'s. By construction, $\operatorname{tr} Q_I = 0$. It is straightforward to verify that $\operatorname{tr} R^2 Q_I = 0$ and that $\operatorname{tr} Q_I Q_J Q_K = 0$.

[9]We add a subscript to the anomalies a_c and c_c in this section to avoid confusion with the Euler characters a, b, and c.

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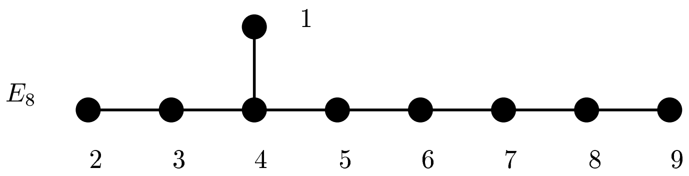

Figure 6: Extended Dynkin diagram of E_8.

We can also investigate the properties of $\operatorname{tr} RQ_IQ_J$. For convenience, we normalize the U(1) charges by dividing out by the rank of the corresponding gauge groups. In particular, bifundamentals with charge ± 1 under a U(1) in the first block will now have charge $\pm 1/xN$, bifundamentals with charge ± 1 under a U(1) in the second block will now have charge $\pm 1/yN$, and bifundamentals with charge ± 1 under a U(1) in the third block will now have charge $\pm 1/zN$. Using the NSVZ beta functions, one finds that $\operatorname{tr} RQ_J^2 = -4$. Moreover, for Q_I and Q_J in the same block $\operatorname{tr} RQ_IQ_J = 2$, if I and J have a node in common. Otherwise, the trace vanishes. One important point is that the matrix $\operatorname{tr} RQ_IQ_J$ has all negative eigenvalues. From [17], we conclude that our choice of R symmetry maximizes $\operatorname{tr} R^3$ over the space of Q_I.

It is amusing to see that the Dynkin diagrams of E_n emerge in a natural way out of these three block collections by using the bases of Q_I and their intersection given in the previous paragraph. Recall that the Q_I span the orthogonal complement of the R-charge in the charge lattice, which is known to be isomorphic to the root lattice of the exceptional Lie algebras.[10] We can make this isomorphism very explicit for the three block collections. To this end, consider the extended Dynkin diagrams of E_n, and search for nodes such that the removal of this node leads to diagrams consisting of (at most) three disconnected A-type Dynkin diagrams, i.e., single lines of nodes. The resulting nodes correspond to a basis of the root lattice of E_n. These bases can be mapped onto the bases of Q_I obtained from three-block collections, which intersect in the same pattern. If one considers all possibilities of removing one node in this way, one recovers exactly the lists of integers (α, β, γ) that characterize the Markov equations of Karpov and Nogin.

For example, consider the extended Dynkin diagram of E_8 in figure 6. Removing node number 4 leads to three groups of nodes with 1, 2, and 5 nodes in each group respectively. This means that $(\alpha, \beta, \gamma) = (2, 3, 6)$ and corresponds to equation number (8.3) on the list of [34]. Similarly, removing node 1 leads to $(\alpha, \beta, \gamma) = (1, 1, 9)$ which is eq. (8.1), removing node 3 gives $(1, 2, 8)$, or eq. (8.2), and, finally, removing node 5 gives $(1, 5, 5)$ or eq. (8.4). In checking the similar statements for the other del Pezzos one has to be careful that where one adds the extending node to the Dynkin diagram of E_n depends on n. For E_3, one has to remove two nodes because there are two extending nodes.

7. The eighth del Pezzo

We crown this paper by applying our technology to the last del Pezzo surface and all its 240 smallest dibaryons.

[10] We recall that by definition, we have $E_5 \cong D_5$, $E_4 \cong A_4$, and $E_3 \cong A_2 \oplus A_1$.

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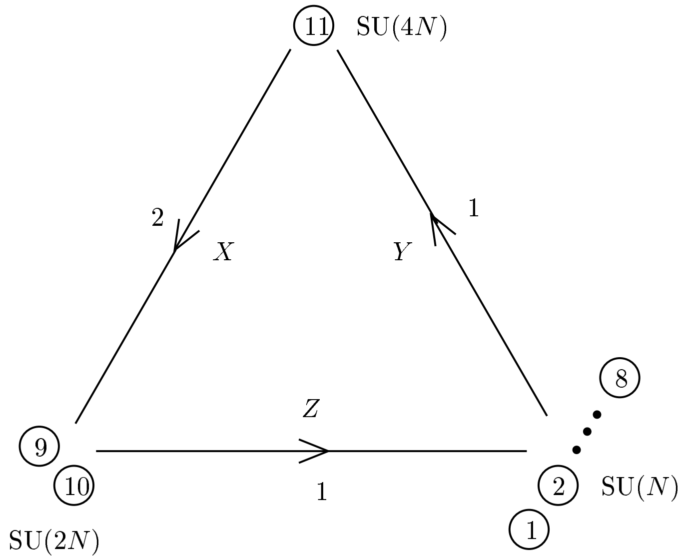

Figure 7: Quiver for dP_8.

To make the job easier, we pick the simplest dual three block collection $(\mathcal{G}^\vee, \mathcal{F}^\vee, \mathcal{E}^\vee)$ that we know about. In particular, we choose the collection labeled (8.2) in the paper by Karpov and Nogin [34]. The first block in the collection contains one node $\mathcal{G}^\vee = (A)$ where A is the right mutation of the trivial bundle $\mathcal{O}$ over the block of exceptional divisors $\mathcal{O}(E_i)$, $i = 4, \ldots, 8$. In particular, A falls into the short exact sequence

$$0 \to \mathcal{O} \to \bigoplus_{i=4}^{8} \mathcal{O}(E_i) \to A \to 0\,. \tag{7.1}$$

Thus $\mathrm{ch}(A) = (4, \sum_{i=4}^{8} E_i, -5/2)$.

The second block has two nodes $\mathcal{F}^\vee = (T, T')$ where T and T' are obtained through left mutations. In particular, T' and T are the left mutations of $\mathcal{O}(H)$ and $\mathcal{O}(2H - E_1 - E_2 - E_3)$ respectively over $\mathcal{O}(H - E_i)$, $i = 1, 2, 3$:

$$0 \to \left\{ \begin{array}{c} T' \\ T \end{array} \right\} \to \bigoplus_{i=1}^{3} \mathcal{O}(H - E_i) \to \left\{ \begin{array}{c} \mathcal{O}(H) \\ \mathcal{O}(2H - E_1 - E_2 - E_3) \end{array} \right\} \to 0\,. \tag{7.2}$$

The charges are $\mathrm{ch}(T) = (2, H, -1/2)$ and $\mathrm{ch}(T') = (2, 2H - E_1 - E_2 - E_3, -1/2)$. The third block contains only line bundles. In particular

$$\mathcal{E}^\vee = \left(\{\mathcal{O}(E_i - K)\}_{i=4,\ldots 8}\,, \{\mathcal{O}(H - E_j)\}_{j=1,2,3} \right)\,. \tag{7.3}$$

We label the bifundamental fields opposite from the third, second, and first blocks X, Y, and Z respectively. From either the gauge theory computation (6.10) or the geometric computation (4.12), we know the R-charges of these fields are $R_X = 1/2$, $R_Y = 1/2$, and $R_Z = 1$. In the interest of keeping things as simple as possible for this complicated example, we restrict attention to the smallest dibaryons, i.e. the dibaryons with R-charge $2N$ or equivalently curves in the del Pezzo with degree one.

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There are a number of different possible ways to construct these smallest dibaryons from gauge theory. We go through each possibility and count up the naive number, ignoring classical relations from the superpotential and any possible "quantum" relations (see [15]). We will see that we get too many dibaryons. We then repeat the calculation geometrically and see which dibaryons we over counted and why.

There are naively three dibaryons constructed from antisymmetrizing over $4N$ copies of the X fields. We can choose one of the SU($2N$) gauge groups twice or each SU($2N$) gauge group once. Geometrically, the divisors for these three dibaryons are

$$\begin{aligned} 2c_1(T) - c_1(A) &= 2H - \sum_{i=4}^{8} E_i\,, \\ 2c_1(T') - c_1(A) &= 4H - 2E_1 - 2E_2 - 2E_3 - \sum_{i=4}^{8} E_i\,, \\ c_1(T) + c_1(T') - c_1(A) &= 3H - \sum_{i=1}^{8} E_i = -K\,. \end{aligned}$$

The first two divisors have genus zero, as can be seen from the genus formula $K \cdot (K + C) = 2g - 2$. The last divisor, which is nothing but the anticanonical bundle, has genus one.

We move onto the dibaryons constructed from antisymmetrizing over N copies of the YXZ fields. As there are eight nodes in the last block, there are 64 possible dibaryons. Geometrically, to construct the divisor, we take the difference of two of the nodes and add $-K$:

$$D_{YXZ} = c_1(E^\vee_{h(YXZ)}) - c_1(E^\vee_{t(YXZ)}) - K\,. \tag{7.4}$$

We see immediately that in the case that $t = h$, $D_{YXZ} = -K$. Thus eight of these divisors are identical and have genus one. So the gauge theory, without additional relations, over counts by seven. The remaining 56 divisors have genus zero, are all distinct, and are summarized in (7.6). Only the genus zero curves are tabulated.

Next we consider the dibaryons, constructed from antisymmetrizing over $2N$ Z type bifundamentals. There are 56 such objects. One might have thought it possible to antisymmetrize over two copies of the same SU(N) gauge group. However, since there is only one Z type bifundamental between any two nodes in the $\mathcal{F}^\vee$ and $\mathcal{E}^\vee$ blocks, such a dibaryon antisymmetrizes to give zero. Thus, we need to make sure the SU(N) gauge groups are distinct.

Geometrically, we find exactly 56 Z type divisors with genus zero and degree one, as tabulated in (7.6). One can ask what the analog of taking two identical SU(N) gauge groups is. For example, consider the divisor

$$2(H - E_1) - H = H - 2E_1 \tag{7.5}$$

obtained by taking two copies of $\mathcal{O}(H - E_1)$ in the third block. This divisor does indeed have degree one, leading to a dibaryon with R-charge $2N$. However, the genus is -1! We conclude there is no such cycle for a D3-brane to wrap.

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There are 70 dibaryons formed from antisymmetrizing over $4N$ copies of the Y type bifundamentals. Similar to the Z case, we need to make sure that all the SU(N) gauge groups are distinct. Geometrically we find 70 distinct genus zero, degree one curves, formed by taking the difference between $c_1(A)$ and the first Chern classes of four distinct bundles in $\mathcal{E}^\vee$ (plus $-K$). The results are tabulated in (7.6).

Finally, we consider the dibaryons of type YX. Geometrically, the divisors are related to the divisors of the Z type dibaryons via $D \to D' = -2K - D$. It is easy to check that if the degree of D is one, then the degree of D' is also one and moreover that D and D' have the same genus. With a little more work to make sure that the D' are distinct from the D, we find the 56 additional degree one, genus zero dibaryons in table 7.6. These dibaryons are obtained by antisymmetrizing over $2N$ copies of the YX fields, making sure that the antisymmetrization is over distinct SU(N) gauge fields. The requirement on the gauge theory side that the SU(N) groups be distinct is presumably enforced by the superpotential or additional "quantum" relations. The argument provided for the Z type dibaryons fails here.

Without a superpotential or knowledge of the additional quantum relations we would have over counted the YXZ and YX type dibaryons. However, by identifying the bifundamental fields with fractional divisors, it becomes clear which gauge invariant combinations of the bifundamental fields correspond to which holomorphic curves.

	number	X	Y	YX	Z	YXZ
E_i	8	0	5	0	3	0
$H - E_i - E_j$	28	0	0	10	3	15
$2H - \sum_{i=1}^{5} E_{a_i}$	56	1	30	10	15	0
$3H - 2E_a - \sum_{i=1}^{6} E_{a_i}$	56	0	0	15	15	26
$4H - 2E_a - 2E_b - 2E_c - \sum_{i=1}^{5} E_{a_i}$	56	1	30	15	10	0
$5H - 2\sum_{i=1}^{6} E_{a_i} - \sum_{j=1}^{2} E_{a_j}$	28	0	0	3	10	15
$6H - 3E_a - 2\sum_{i=1}^{7} E_{a_i}$	8	0	5	3	0	0
totals	240	2	70	56	56	56

(7.6)

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